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Yetter–Drinfel'd categories associated to an arbitrary bialgebra

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Abstract

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Various prebraided monoidal categories associated to a bialgebra over a commutative ring are studied and their relationships at various levels are examined. Generalizations of braided bialgebras are described and prebraided monoidal categories are associated with them. Three formally different braided monoidal categories can always be associated with any bialgebra over a commutative ring. These are not necessarily the same.

1. Introduction

Braided categories have been found to underlie many applications of quantum groups, especially to low-dimensional topology and to conformal field theory. Such braided categories first arose as categories of modules over special Hopf algebras constructed by Faddeev, Reshetikhin and Takhtajan. Later, Drinfel'd gave a more general construction via the celebrated Drinfel'd double of a finite-dimensional Hopf algebra. (There seem to be difficulties, if we wish to drop

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the hypothesis that A is finite-dimensional. Much of the existing literature leaves unexamined the topological ramifications involved in this matter.)

The present paper attempts to study Yetter's category construction, along lines which will be discussed below. We note that (although this fact is not explicitly spelled out in Yetter's paper) Yetter's categories seem to include all those mentioned above:

(1) Majid, in his paper [12] cites [18] and shows that if A is a finite-dimensional Hopf algebra, the braided category constructed from A by Drinfel'd's double construction (suitably redefined) indeed coincides with one of Yetter's categories.

(2) In [8] it is proved that the FRT constructions are a special case of the category of comodules over braided bialgebras A, defined in [8, Definition 2.2] which are there proved to be prebraided categories (and braided if A is a Hopf algebra) [8, Theorem 2.7]. In the present paper we show this is also a special case of Yetter's construction.

(3) Results which imply the equivalence between the FRT categories and the appropriate special cases of Yetter's construction are established (independently of (2)) in [14].

We shall refer to the category constructed by Yetter (together with prebraiding structure) over the base bialgebra A as the *Yetter–Drinfel'd category* ${}_A^A\mathcal{YD} = {}_A\mathcal{YD}$. (In Yetter's notation this is denoted by A–**cbm**; Yetter shows this category has a natural prebraiding, which is a braiding if A^{cop} is a Hopf algebra.)

In this paper we study the category ${}_A^A\mathcal{YD}$ and its formal variations, notably ${}_A\mathcal{YD}^A$, and focus on invertibility questions pertaining to the prebraiding maps $\sigma_{M,N}$ defined for ${}_A\mathcal{YD}^A$ which are analogs of the $s_{X,Y}$ defined in [18]. We are particularly interested in the case when M and N are finite-dimensional.

We can *always* associate braided categories with a bialgebra by Theorem 11. These are full sub-prebraided categories of ${}_A\mathcal{YD}^A$ which are roughly analogous to the left invertible elements, the right invertible elements and invertible elements respectively of a semigroup. Generally these subcategories are different.

The paper is organized as follows. We review basic facts about coalgebras and the Yang–Baxter equation in Section 2. In Section 3 we review the notion of Yetter–Drinfel'd structure, describe its left–right variations and give examples of where they arise.

In Section 4 we generalize the concept of braided bialgebra as defined in [8] to the notion of weakly braided bialgebra A and show that every left A-comodule has in a natural way a Yetter–Drinfel'd structure. We also show that not all Yetter–Drinfel'd structures over A arise in this way.

In Section 5 we define the concept of *left* (*right*) *invertible* Yetter–Drinfel'd structure, and construct a braided structure on the full subcategories constituted by the left invertible, right invertible and invertible objects respectively. In Section 6 we show that ${}_A^A\mathcal{YD}$ has two natural prebraiding structures and relate them.

Finally, in Section 7 we construct a number of counterexamples, and study the question of when the prebraiding map

$$\sigma_{M,N} : M \otimes N \to N \otimes M$$

is invertible (when M and N are finite-dimensional).

2. Preliminaries

Throughout this paper k is a commutative ring with unity. The tensor product of k-modules is taken over the ground ring k. For k-modules U and V, we define $T_{U,V} : U \otimes V \to V \otimes U$ by $T_{U,V}(u \otimes v) = v \otimes u$ for $u \in U$ and $v \in V$.

Suppose that (C, Δ, ε) is a coalgebra over k. Then $g \in C$ is said to be a *grouplike element* if $\Delta(g) = g \otimes g$ and $\varepsilon(g) = 1$. The set of grouplike elements of C is denoted by $G(C)$. The linear dual $C^* = \mathrm{Hom}_k(C, k)$ of C is an algebra, with multiplication given by $f * g(c) = \sum f(c_{(1)})g(c_{(2)})$ for $f, g \in C^*$ and $c \in C$ and unity ε.

Now suppose that (M, ρ) is a C-comodule. For $m \in M$ we modify the Heyneman–Sweedler notation for expressing $\rho(m)$ by writing $\rho(m) = \sum m_{\langle 1 \rangle} \otimes m_{(2)} \in M \otimes C$ when M is a right C-comodule, and $\rho(m) = \sum m_{(1)} \otimes m_{\langle 2 \rangle} \in C \otimes M$ when M is a left C-comodule. We will be dealing with complicated expressions involving several kinds of products and coproducts. We assume throughout this paper that all other operators take precedence over tensor product.

The 'opposite' coalgebra C^{cop} is C as a k-module with comultiplication given by $\Delta^{\mathrm{cop}}(c) = \sum c_{(2)} \otimes c_{(1)}$ for $c \in C$, where we write $\Delta(c) = \sum c_{(1)} \otimes c_{(2)}$. If A is an algebra over k, then the 'opposite' algebra A^{op} is A as a k-module with multiplication defined by $a \cdot b = ba$ for $a, b \in A$. The reader should note that when A is a bialgebra A^{cop} and A^{op}, and hence $A^{\mathrm{cop\,op}}$, are bialgebras.

We will need to make use of 'opposite' actions in Section 3. Suppose that C is a coalgebra over k, and that (M, ρ) is a left (respectively right) C-module. Then (M, ρ^{cop}) is a right (respectively left) C^{cop}-comodule where $\rho^{\mathrm{cop}}(m) = \sum m_{\langle 2 \rangle} \otimes m_{(1)}$ (respectively $\rho^{\mathrm{cop}}(m) = \sum m_{(2)} \otimes m_{\langle 1 \rangle}$) for $m \in M$. Now let A be an algebra over k, and suppose that $(M, \cdot)$ is a left (respectively right) A-module. Then $(M, \cdot^{\mathrm{op}})$ is a right (respectively left) A^{op}-module, where $m \cdot^{\mathrm{op}} a = a \cdot m$ (respectively $a \cdot^{\mathrm{op}} m = m \cdot a$) for $m \in M$ and $a \in A$. We recommend [15] for a discussion of and basic results for coalgebras, bialgebras and Hopf algebras.

Now let A be a bialgebra over k. Let M be a right A-comodule and N be a left A-module. Define $\sigma_{M,N} : M \otimes N \to N \otimes M$ by

$$\sigma_{M,N}(m \otimes n) = \sum m_{(2)} \cdot n \otimes m_{\langle 1 \rangle} \tag{1}$$

for $m \in M$ and $n \in N$.

Lemma 1. *Suppose that A is a Hopf algebra with antipode s over a commutative ring k. Let M be a right A-comodule and let N be a left A-module. Then $\sigma_{M,N}$ is an invertible k-linear map.*

Proof. The lemma is very similar to the technical part of [18, Theorem 7.2]. We give a different argument, following the proof of part (a) of [14, Proposition 2].

Define $\tau_{M,N} : N\otimes M \to M\otimes N$ by $\tau_{M,N}(n\otimes m) = \sum m_{\langle 1\rangle}\otimes s(m_{(2)})\cdot n$ for $n\in N$ and $m\in M$. Observe that

$$\begin{aligned}
&\tau_{M,N}\circ\sigma_{M,N}(m\otimes n)\\
&\quad = \tau_{M,N}\Big(\sum m_{(2)}\cdot n\otimes m_{\langle 1\rangle}\Big)\\
&\quad = \sum m_{\langle 1\rangle\langle 1\rangle}\otimes s(m_{\langle 1\rangle(2)})\cdot(m_{(2)}\cdot n)\\
&\quad = \sum m_{\langle 1\rangle}\otimes (s(m_{(2)(1)})m_{(2)(2)})\cdot n\\
&\quad = \sum m_{\langle 1\rangle}\otimes \varepsilon(m_{(2)})n\\
&\quad = m\otimes n\,,
\end{aligned}$$

and likewise $\sigma_{M,N}\circ\tau_{M,N}(n\otimes m) = n\otimes m$. $\square$

Definition. Let V be a k-module. A k-linear map $R: V\otimes V\to V\otimes V$ is said to satisfy the *Yang–Baxter* condition if

$$(R\otimes 1_V)\circ(1_V\otimes R)\circ(R\otimes 1_V) = (1_V\otimes R)\circ(R\otimes 1_V)\circ(1_V\otimes R)\,. \tag{2}$$

Remark. The classical example of an operator which satisfies (2) is

$$T_V = T_{V,V} : V\otimes V\to V\otimes V\,.$$

While some authors [8, 17] prefer the formulation (2), others [2, 3, 7] refer to the condition

$$R_{12}\circ R_{13}\circ R_{23} = R_{23}\circ R_{13}\circ R_{12} \tag{3}$$

as the Yang–Baxter condition, where $R_{12} = R\otimes 1_V$, $R_{23} = 1_V\otimes R$ and $R_{13} = (1_V\otimes T_V)\circ(R\otimes 1_V)\circ(1_V\otimes T_V)$. The choice of convention in this matter is simply a matter of convenience, since R is a solution to one if and only if $R\circ T_V$ (or equivalently $T_V\circ R$) is a solution to the other. See [8, Remark 4.3] for a more detailed discussion of this equivalence. Note also that if R satisfies (2) so does $T_V\circ R\circ T_V$. In the present paper we prefer (2) to (3) since (2) seems more directly related to braided structures which will here be studied.

Some readers will be more familiar with (2) in the form

$$R_{12} \circ R_{23} \circ R_{12} = R_{23} \circ R_{12} \circ R_{23} . \tag{4}$$

3. Yetter–Drinfel'd structures

Throughout this section, we assume that k is a commutative ring and A is a bialgebra over k.

Suppose that M simultaneously has a left A-module structure $(M, \cdot)$ and a left A-comodule structure (M, ρ). Then $(M, \cdot, \rho)$ is called a *left Hopf module* [15] if the two structures are compatible in the sense that

$$\rho(a \cdot m) = \sum a_{(1)} m_{(1)} \otimes a_{(2)} \cdot m_{\langle 2 \rangle}$$

for all $a \in A$ and $m \in M$. In this paper we are concerned with a very different compatibility condition.

Definition 2. A *left–left Yetter–Drinfel'd structure over* A is a triple $(M, \cdot, \rho)$, where $(M, \cdot)$ is a left A-module and (M, ρ) is a left A-comodule such that the compatibility condition

$$\sum a_{(1)} m_{(1)} \otimes a_{(2)} \cdot m_{\langle 2 \rangle} = \sum (a_{(1)} \cdot m)_{(1)} a_{(2)} \otimes (a_{(1)} \cdot m)_{\langle 2 \rangle} \tag{5}$$

is satisfied for all $a \in A$ and $m \in M$.

These structures are the objects of a category, which we shall denote by ${}^{A}_{A}\mathcal{YD}$, whose morphisms $f : (M, \cdot, \rho) \to (M, \cdot', \rho')$ are maps $f : M \to M'$ which are simultaneously module and comodule maps.

Proposition 3. *Suppose that* $(M, \cdot, \rho)$ *is a left–left Yetter–Drinfel'd structure over* A. *Then the operator* $R_M : M \otimes M \to M \otimes M$ *defined by*

$$R_M(m \otimes n) = \sum m_{(1)} \cdot n \otimes m_{\langle 2 \rangle}$$

satisfies the Yang–Baxter condition. If A^{cop} *is a Hopf algebra, then* R_M *is invertible.*

Proof. For a diagrammatic proof see [18]. Let $l,m,n \in M$. An immediate calculation gives

$$\begin{aligned}
&(R_M \otimes 1_M) \circ (1_M \otimes R_M) \circ (R_M \otimes 1_M)(l \otimes m \otimes n) \\
&\quad = \sum (l_{(1)} \cdot m)_{(1)} \cdot (l_{\langle 2 \rangle (1)} \cdot n) \otimes (l_{(1)} \cdot m)_{\langle 2 \rangle} \otimes l_{\langle 2 \rangle \langle 2 \rangle} \\
&\quad = \sum ((l_{(1)(1)} \cdot m)_{(1)} l_{(1)(2)}) \cdot n \otimes (l_{(1)(1)} \cdot m)_{\langle 2 \rangle} \otimes l_{\langle 2 \rangle}
\end{aligned}$$

and

$$
\begin{aligned}
&(1_M \otimes R_M) \circ (R_M \otimes 1_M) \circ (1_M \otimes R_M)(l \otimes m \otimes n) \\
&\quad = \sum l_{(1)} \cdot (m_{(1)} \cdot n) \otimes l_{\langle 2 \rangle (1)} \cdot m_{\langle 2 \rangle} \otimes l_{\langle 2 \rangle \langle 2 \rangle} \\
&\quad = \sum (l_{(1)(1)} \cdot m_{(1)}) \cdot n \otimes l_{(1)(2)} \cdot m_{\langle 2 \rangle} \otimes l_{\langle 2 \rangle} .
\end{aligned}
$$

Thus R_M satisfies the Yang–Baxter condition. If A^{cop} is a Hopf algebra, then R_M is invertible by [18, Theorem 7.2]. This concludes the proof. $\square$

Remark 4. If M is a finitely generated free k-module, then Proposition 2 accounts for all $R : M \otimes M \to M \otimes M$ satisfying the Yang–Baxter condition in the following sense: M has a left–left Yetter–Drinfel'd structure over the bialgebra $A(R \circ T_M)^{\mathrm{cop}}$ such that $R = R_M$ [7, Proposition 3.3.1].

There are three obvious variations on Definition 2, pairing left or right module structures with left or right comodule structures. Thus we are led to three other categories (in the following $a \in A$ and $m \in M$): the category ${}_A\mathcal{YD}^A$ of *left–right* Yetter–Drinfel'd structures $(M, \cdot, \rho)$, where $(M, \cdot)$ is a left A-module and (M, ρ) is a right A-comodule satisfying

$$
\sum a_{(1)} \cdot m_{\langle 1 \rangle} \otimes a_{(2)} m_{(2)} = \sum (a_{(2)} \cdot m)_{\langle 1 \rangle} \otimes (a_{(2)} \cdot m)_{(2)} a_{(1)} ; \tag{6}
$$

the category ${}^A\mathcal{YD}_A$ of *right–left* Yetter–Drinfel'd structures $(M, \cdot, \rho)$, where $(M, \cdot)$ is a right A-module and (M, ρ) is a left A-comodule satisfying

$$
\sum m_{(1)} a_{(1)} \otimes m_{\langle 2 \rangle} \cdot a_{(2)} = \sum a_{(2)} (m \cdot a_{(1)})_{(1)} \otimes (m \cdot a_{(1)})_{\langle 2 \rangle} ; \tag{7}
$$

the category $\mathcal{YD}_A^A$ of *right–right* Yetter–Drinfel'd structures $(M, \cdot, \rho)$, where $(M, \cdot)$ is a right A-module and (M, ρ) is a right A-comodule satisfying

$$
\sum m_{\langle 1 \rangle} \cdot a_{(1)} \otimes m_{(2)} a_{(2)} = \sum (m \cdot a_{(2)})_{\langle 1 \rangle} \otimes a_{(1)} (m \cdot a_{(2)})_{(2)} . \tag{8}
$$

These four types of Yetter–Drinfel'd structures are equivalent in the formal sense that for any bialgebra A

$$
{}_A^A\mathcal{YD} \simeq {}_B\mathcal{YD}^B \simeq {}^C\mathcal{YD}_C \simeq \mathcal{YD}_D^D
$$

where $B = A^{\mathrm{cop}}$, $C = A^{\mathrm{op}}$ and $D = A^{\mathrm{cop\,op}}$.

Remark 5. For a given bialgebra A, it is not clear how ${}_A^A\mathcal{YD}$, ${}_A\mathcal{YD}^A$, ${}^A\mathcal{YD}_A$ and $\mathcal{YD}_A^A$ are related on other levels.

We can say, however, that when A has bijective antipode s, they are the same categories. Note that s is an algebra and a coalgebra anti-endomorphism (see [15, Proposition 4.0.1]). For a right A-comodule (M, ρ), let (M, ρ_s) denote the left A-comodule structure on M derived from (M, ρ) by $\rho_s(m) = \sum s(m_{(2)}) \otimes m_{\langle 1 \rangle}$ for $m \in M$. For a right A-module $(M, \cdot)$, let $(M, \cdot_{s^{-1}})$ denote the left A-module structure on M derived from $(M, \cdot)$ by $a \cdot_{s^{-1}} m = m \cdot s^{-1}(a)$ for $a \in A$ and $m \in M$. For left comodules and modules we use the same notation for the analogous right objects.

Proposition 6. *Suppose that A is a bialgebra with bijective antipode s over a commutative ring k. Then*:

(a) $(M, \cdot, \rho) \mapsto (M, \cdot_{s^{-1}}, \rho_s)$ *and* $f \mapsto f$ *describes categorical isomorphisms*

$$\mathcal{YD}_A^A \simeq {}_A^A\mathcal{YD} \quad \text{and} \quad {}_A\mathcal{YD}^A \simeq {}^A\mathcal{YD}_A\,.$$

(b) $(M, \cdot, \rho) \mapsto (M, \cdot, \rho_s)$ *and* $f \mapsto f$ *describes a categorical isomorphism*

$${}_A\mathcal{YD}^A \simeq {}_A^A\mathcal{YD}\,.$$

Proof. Part (a) follows by [7, Proposition 4.5.1]. To see part (b), first assume that $(M, \cdot, \rho)$ is an object of ${}_A\mathcal{YD}^A$. The calculation

$$\begin{aligned}
&\sum a_{(1)}s(m_{(2)}) \otimes a_{(2)} \cdot m_{\langle 1 \rangle} \\
&\quad = \sum a_{(1)}s(m_{(2)})\varepsilon(a_{(3)}) \otimes a_{(2)} \cdot m_{\langle 1 \rangle} \\
&\quad = \sum a_{(1)}s(m_{(2)})s(a_{(3)})a_{(4)} \otimes a_{(2)} \cdot m_{\langle 1 \rangle} \\
&\quad = \sum a_{(1)}s((a_{(3)} \cdot m)_{(2)}a_{(2)})a_{(4)} \otimes (a_{(3)} \cdot m)_{\langle 1 \rangle} \\
&\quad = \sum a_{(1)}s(a_{(2)})s((a_{(3)} \cdot m)_{(2)})a_{(4)} \otimes (a_{(3)} \cdot m)_{\langle 1 \rangle} \\
&\quad = \sum \varepsilon(a_{(1)})s((a_{(2)} \cdot m)_{(2)})a_{(3)} \otimes (a_{(2)} \cdot m)_{\langle 1 \rangle} \\
&\quad = \sum s((a_{(1)} \cdot m)_{(2)})a_{(2)} \otimes (a_{(1)} \cdot m)_{\langle 1 \rangle}
\end{aligned}$$

for all $a \in A$ and $m \in M$ shows that $(M, \cdot, \rho_s)$ is an object of ${}_A^A\mathcal{YD}$. The remaining details of the proof of part (b) are left to the reader. □

Remark 7. Let A be a finite-dimensional Hopf algebra over a field k. Suppose $D(A)$ is the Drinfel'd double of A as defined in [2] and $D'(A)$ is the Drinfel'd double of A as defined by Majid in [12]. Majid notes that $I \otimes S : D'(A) \rightarrow D(A)$ is a Hopf algebra isomorphism, where S is the antipode of A^*. He shows that there is a categorical isomorphism

$${}_{D'(A)}\mathcal{M} \simeq {}_A^A\mathcal{YD}\,.$$

By [14, Proposition 2.4] it follows that there is a categorical isomorphism

$$_{D(A)}\mathcal{M} \simeq {}_{A}\mathcal{YD}^{A} .$$

The objects of $_{A}\mathcal{YD}^{A}$ are called left quantum Yang–Baxter A-modules in [7, 14], and the category $_{A}\mathcal{YD}^{A}$ is the category $_{A}\mathcal{YB}$ of [7].

4. Right, left weakly braided bialgebras

In [10, Section 2] braided bialgebras over a commutative ring k are defined and studied, particularly from the point of view of the category of their right comodules. We show in Proposition 8 that the right A-comodules of a braided bialgebra A have a natural left A-module structure which makes them objects of $_{A}\mathcal{YD}^{A}$, and we show in Proposition 9 that the left A-comodules have a natural left A-module structure which makes them objects of $^{A}_{A}\mathcal{YD}$. Propositions 8 and 9 are true for more extensive classes of bialgebras.

Let A be a bialgebra over k, and suppose that $\langle \mid \rangle : A \times A \to k$ is a k-bilinear form. We say that $\langle \mid \rangle$ is a *weak braiding structure on* A, or that the pair $(A, \langle \mid \rangle)$ is a *weakly braided bialgebra* if

(WB1) $\quad \sum \langle a_{(1)} | b_{(1)} \rangle b_{(2)} a_{(2)} = \sum a_{(1)} b_{(1)} \langle a_{(2)} | b_{(2)} \rangle$,

(WB2) $\quad \langle a | 1 \rangle = \varepsilon(a)$,

(WB3) $\quad \langle a | bc \rangle = \sum \langle a_{(1)} | b \rangle \langle a_{(2)} | c \rangle$,

(WB4) $\quad \langle 1 | a \rangle = \varepsilon(a)$,

(WB5) $\quad \langle ab | c \rangle = \sum \langle a | c_{(2)} \rangle \langle b | c_{(1)} \rangle$

hold for all $a,b,c \in A$. By [8, Theorem 2.7] it follows that $(A, \langle \mid \rangle)$ is a braided bialgebra if and only if (WB1)–(WB5) are satisfied and $\langle \mid \rangle$ is invertible. If (WB1)–(WB3) are satisfied we say that $\langle \mid \rangle$ is a *right* weak braiding structure on A, or that the pair $(A, \langle \mid \rangle)$ is a *right* weakly braided bialgebra.

Conditions (WB2)–(WB5) have a natural formulation in terms of algebra homomorphisms. Let $\beta = \langle \mid \rangle$ and define $\beta_\ell, \beta_r : A \to A^*$ by $\beta_\ell(a)(b) = \beta(a, b) = \beta_r(b)(a)$ for $a,b \in A$. Then (WB2) and (WB3) are satisfied if and only if $\beta_r : A \to A^*$ is an algebra homomorphism. Likewise (WB4) and (WB5) are satisfied if and only if $\beta_\ell : A \to A^{*\mathrm{op}}$ is an algebra homomorphism.

A weakly braided bialgebra is not necessarily braided. For example, let $S = \{1, e\}$ be the two-element semigroup for which 1 is the identity element and $e^2 = e$. Then the semigroup algebra $A = k[S]$ of S over k has a weak braiding structure $\langle \mid \rangle$ determined by $\langle e | e \rangle = 0$ which is not a braiding structure by [8, Theorem 2.5].

The right comodules of a right weakly braided bialgebra A have a natural left

A-module structure which makes them objects of ${}_A\mathcal{YD}^A$. Let $(A, \langle \mid \rangle)$ be a right weakly braided bialgebra, and suppose that (M, ρ) is a right A-comodule. Then M has a (rational) left A^*-module structure defined by $a^* \cdot m = \sum m_{\langle 1 \rangle} a^*(m_{(2)})$ for $a^* \in A^*$ and $m \in M$. Thus M has a left A-module structure by pull-back along the algebra homomorphism β_r. The action is described by $a \cdot m = \sum m_{\langle 1 \rangle} \langle m_{(2)} | a \rangle$ for $a \in A$ and $m \in M$. The category of right A-modules may be regarded as a subcategory of ${}_A\mathcal{YD}^A$. This assertion is justified for the most part by the next proposition.

Proposition 8. *Suppose that $(A, \langle \mid \rangle)$ is a right weakly braided bialgebra over a commutative ring k. Let (M, ρ) be a right A-comodule and let $a \cdot m = \sum m_{\langle 1 \rangle} \langle m_{(2)} | a \rangle$ for $a \in A$ and $m \in M$. Then:*

(a) *$(M, \cdot)$ is a left A-module.*

(b) *$(M, \cdot, \rho)$ is an object of ${}_A\mathcal{YD}^A$.*

Proof. We need only show part (b). To do this we calculate

$$
\begin{aligned}
&\sum a_{(1)} \cdot m_{\langle 1 \rangle} \otimes a_{(2)} m_{(2)} \\
&\quad = \sum m_{\langle 1 \rangle \langle 1 \rangle} \langle m_{\langle 1 \rangle (2)} | a_{(1)} \rangle \otimes a_{(2)} m_{(2)} \\
&\quad = \sum m_{\langle 1 \rangle} \otimes \langle m_{(2)(1)} | a_{(1)} \rangle a_{(2)} m_{(2)(2)}
\end{aligned}
$$

and

$$
\begin{aligned}
&\sum (a_{(2)} \cdot m)_{\langle 1 \rangle} \otimes (a_{(2)} \cdot m)_{(2)} a_{(1)} \\
&\quad = \sum m_{\langle 1 \rangle \langle 1 \rangle} \otimes \langle m_{(2)} | a_{(2)} \rangle m_{\langle 1 \rangle (2)} a_{(1)} \\
&\quad = \sum m_{\langle 1 \rangle} \otimes \langle m_{(2)(2)} | a_{(2)} \rangle m_{(2)(1)} a_{(1)} \\
&\quad = \sum m_{\langle 1 \rangle} \otimes m_{(2)(1)} a_{(1)} \langle m_{(2)(2)} | a_{(2)} \rangle
\end{aligned}
$$

for all $a \in A$ and $m \in M$. Therefore part (b) follows by (WB1). This concludes our proof. $\square$

Proposition 8 has an analog for left comodules of a left weakly braided bialgebra. Let A be a bialgebra over k, and suppose that $\langle \mid \rangle : A \times A \to k$ is a k-bilinear form. We say that $(A, \langle \mid \rangle)$ is a *left* weakly braided bialgebra if (WB1), (WB4) and (WB5) are satisfied.

Suppose that $(A, \langle \mid \rangle)$ is a left weakly braided bialgebra over k, and let (M, ρ) be a left A-comodule. Then M has a (rational) right A^*-module structure defined by $m \cdot a^* = \sum a^*(m_{(1)}) m_{\langle 2 \rangle}$ for $m \in M$ and $a \in A^*$. Therefore, M has a left A-module structure by pull-back along the algebra anti-homomorphism β_ℓ, which is thus described by $a \cdot m = \sum \langle a | m_{(1)} \rangle m_{\langle 2 \rangle}$ for $a \in A$ and $m \in M$. The category of left A-comodules can be regarded as a subcategory of ${}_A^A\mathcal{YD}$. This fact is established in part by the following analog of Proposition 8, the proof of which is left to the reader.

Proposition 9. *Suppose that $(A, \langle\ |\ \rangle)$ is a left weakly braided bialgebra over a commutative ring k. Let (M, ρ) be a left A-comodule and let $a \cdot m = \sum \langle a | m_{(1)} \rangle m_{\langle 2 \rangle}$ for $a \in A$ and $m \in M$. Then:*

(a) *$(M, \cdot)$ is a left A-module,*

(b) *$(M, \cdot, \rho)$ is an object of ${}_A^A\mathcal{YD}$.* □

Let $(A, \langle\ |\ \rangle)$ be a right (respectively left) weakly braided bialgebra over a field k, and suppose that (M, ρ) is a right (respectively left) A-comodule. Regard $(M, \cdot, \rho)$ as the object of ${}_A\mathcal{YD}^A$ (respectively ${}_A^A\mathcal{YD}$) described in Proposition 8 (respectively Proposition 9). Then $(M, \cdot)$ is *locally finite*; that is, every finite-dimensional subspace of M is contained in a finite-dimensional sub-(co)module of M, which is therefore an object of ${}_A\mathcal{YD}^A$ (respectively ${}_A^A\mathcal{YD}$).

Generally the module structure of objects of ${}_A\mathcal{YD}^A$ or ${}_A^A\mathcal{YD}$ is not locally finite. For there are ways of turning A into an object of ${}_A\mathcal{YD}^A$, where A is a Hopf algebra with bijective antipode s over a field, and the module action is multiplication [7, Section 8]. Noting that ${}_A^A\mathcal{YD} = {}_{A^{\mathrm{cop}}}\mathcal{YD}^{A^{\mathrm{cop}}}$, and that s^{-1} is the antipode of A^{cop}, we see that the following holds:

Proposition 10. *Suppose that A is a Hopf algebra with bijective antipode s over a field k. Then the following are equivalent:*

(a) *The module structure of all objects M of ${}_A^A\mathcal{YD}$ is locally finite.*

(b) *The module structure of all objects M of ${}_A\mathcal{YD}^A$ is locally finite.*

(c) *A is finite-dimensional.* □

5. A monoidal structure on ${}_A\mathcal{YD}^A$ and certain subcategories

Let A be a bialgebra over k, and suppose that L, M and N are objects of ${}_A\mathcal{YD}^A$. Then by part (b) of [7, Proposition 4.3.1], for example, it follows that $M \otimes N$ is an object of ${}_A\mathcal{YD}^A$, where

$$a \cdot (m \otimes n) = \sum a_{(2)} \cdot m \otimes a_{(1)} \cdot n ,$$

and

$$\rho(m \otimes n) = \sum (m_{\langle 1 \rangle} \otimes n_{\langle 1 \rangle}) \otimes m_{(2)} n_{(2)}$$

for all $a \in A$, $m \in M$ and $n \in N$. Note the 'twist' in the definition of the module action on $M \otimes N$.

Let $\mathbf{I} = k$. Applying [18, Theorem 5.2] to ${}_A\mathcal{YD}^A \simeq {}^{A^{\mathrm{cop}}}_{\mathcal{A}^{\mathrm{cop}}}\mathcal{YD}$, it follows that $({}_A\mathcal{YD}^A, \otimes, \mathbf{I}, \alpha, \rho, \lambda)$ is a monoidal category [4, Section 1], $a \cdot 1 = \varepsilon(a)1$ for $a \in A$ and $\rho(1) = 1 \otimes 1$, and $\alpha_{L,M,N} : (L \otimes M) \otimes N \cong L \otimes (M \otimes N)$, $\rho_L : L \otimes \mathbf{I} \cong L$ and $\lambda_L : \mathbf{I} \otimes L \cong L$ are the natural isomorphisms of the underlying k-modules, which are isomorphisms in the category ${}_A\mathcal{YD}^A$. Note that $\sigma_{M,N}$ defined by

$$\sigma_{M,N}(m \otimes n) = \sum m_{(2)} \cdot n \otimes m_{\langle 1 \rangle}$$

for $m \in M$ and $n \in N$ corresponds under the isomorphism ${}_A\mathcal{YD}^A \simeq {}^{A^{\mathrm{cop}}}_{A^{\mathrm{cop}}}\mathcal{YD}$ to $s_{M,N}$ defined in [18, Section 5].

The identities

$$\sigma_{L,M\otimes N} = (1_M \otimes \sigma_{L,N}) \circ (\sigma_{L,M} \otimes 1_N) \tag{9}$$

and

$$\sigma_{L\otimes M,N} = (\sigma_{L,N} \otimes 1_M) \circ (1_L \otimes \sigma_{M,N}) \tag{10}$$

follow directly from definitions by evaluation on $l \otimes m \otimes n$, where $l \in L$, $m \in M$ and $n \in N$.

Let M be an object of ${}_A\mathcal{YD}^A$. We say that M is *left invertible* (respectively *right invertible*) if $\sigma_{M,N}$ (respectively $\sigma_{N,M}$) is an invertible k-linear map for all objects N of ${}_A\mathcal{YD}^A$. We say that M is *invertible* if M is both left and right invertible. Note that the tensor product of left invertible objects of ${}_A\mathcal{YD}^A$ is left invertible by (10), and that the tensor product of right invertible objects is right invertible by (9). Therefore the tensor product of invertible objects is invertible.

Any k-module V supports an invertible structure. For the module and comodule structures on V defined by

$$a \cdot v = \varepsilon(a)v \quad \text{and} \quad \rho(v) = v \otimes 1$$

respectively for $a \in A$ and $v \in V$ give V the structure of an object of ${}_A\mathcal{YD}^A$ such that $\sigma_{V,M} = T_{V,M}$ and $\sigma_{M,V} = T_{M,V}$ for objects M of ${}_A\mathcal{YD}^A$. We call $(V, \cdot, \rho)$ a *trivial object*.

We let $({}_A\mathcal{YD}^A)^{\lhd}$ (respectively $({}_A\mathcal{YD}^A)^{\rhd}$, $({}_A\mathcal{YD}^A)^{\bullet}$) denote the full subcategory of ${}_A\mathcal{YD}^A$ whose objects are left invertible (respectively right invertible, invertible) objects of ${}_A\mathcal{YD}^A$. Left invertible does not imply right invertible, and right invertible does not imply left invertible either as we will see by example in Section 7. Thus these categories are different in general. If A is a Hopf algebra, then all are identical and are equal to ${}_A\mathcal{YD}^A$ by Lemma 1.

Let $\mathcal{C}$ denote any of these full subcategories. It is easy to see that k is an object of $\mathcal{C}$. We have observed that $\mathcal{C}$ is closed under tensor product. In particular $(\mathcal{C}, \otimes, \mathbf{I}, \rho, \lambda)$ is a monoidal category, where α, ρ and λ are defined as above on objects of $\mathscr{C}$. Observe that $\mathscr{C}$ is closed under (finite) direct sums. If k is a field, then L and M/L are objects of $\mathscr{C}$ whenever L is a subobject of an object M of $\mathscr{C}$.

6. Braiding structures on certain subcategories of ${}_A\mathscr{YD}^A$

Throughout this section A is a bialgebra over k. Let $\mathscr{C}$ be any one of the categories $({}_A\mathscr{YD}^A)^{\lhd}$, $({}_A\mathscr{YD}^A)^{\rhd}$ and $({}_A\mathscr{YD}^A)^{\bullet}$, and suppose that L, M and N are objects of $\mathscr{C}$. In this section we discuss two braiding structures on $\mathscr{C}$, using [7, Proposition 4.3.1] and following [18].

By [18, Theorem 5.2] applied to ${}_A\mathscr{YD}^A \simeq {}^{A^{\mathrm{cop}}}_{A^{\mathrm{cop}}}\mathscr{YD}$ it follows that the family of $\sigma_{M,N}$'s gives the monoidal category $(\mathscr{C}, \otimes, \mathbf{I}, \alpha, \rho, \lambda)$ a prebraiding structure [4, 6]. That is to say

(B1) $\sigma_{M,N} : M\otimes N \to N\otimes M$ is a morphism in ${}_A\mathscr{YD}^{\mathscr{A}}$, and

(B2) equations (9) and (10) hold.

The hexagonal identities reduce to the equations (9) and (10). This prebraiding structure is a braiding structure since

(B3) $\sigma_{M,N}$ is an invertible k-linear map

by definition.

Theorem 11. *Suppose that A is a bialgebra over a commutative ring k. Let $\mathscr{C}$ be any of the categories $({}_A\mathscr{YD}^A)^{\lhd}$, $({}_A\mathscr{YD}^A)^{\rhd}$ and $({}_A\mathscr{YD}^A)^{\bullet}$. Then the maps $\sigma_{M,N}$ defined by* (1) *for objects M, N of $\mathscr{C}$ determine a braiding structure on the monoidal category $(\mathscr{C}, \otimes, \mathbf{I}, \alpha, \rho, \lambda)$.* $\square$

For the sake of completeness, we give a short argument here for (B1). A diagrammatic proof is found in [18].

To show (B1) we show that $\sigma_{M,N}$ is a left module and a right comodule map for objects M, N of ${}_A\mathscr{YD}^A$. Suppose that $m \in M$ and $n \in N$. Since

$$\begin{aligned}
&\sigma_{M,N}(a\cdot(m\otimes n))\\
&\quad = \sigma_{M,N}\left(\sum a_{(2)}\cdot m\otimes a_{(1)}\cdot n\right)\\
&\quad = \sum (a_{(2)}\cdot m)_{(2)}\cdot(a_{(1)}\cdot n)\otimes(a_{(2)}\cdot m)_{\langle 1\rangle}\\
&\quad = \sum ((a_{(2)}\cdot m)_{(2)}a_{(1)})\cdot n\otimes(a_{(2)}\cdot m)_{\langle 1\rangle}\\
&\quad = \sum a_{(2)}\cdot(m_{(2)}\cdot n)\otimes a_{(1)}\cdot m_{\langle 1\rangle}\\
&\quad = a\cdot(\sigma_{M,N}(m\otimes n)),
\end{aligned}$$

it follows that $\sigma_{M,N}$ is a module map. Since

$$\begin{aligned}
&\rho(\sigma_{M,N}(m\otimes n))\\
&\quad=\rho\Big(\sum m_{(2)}\cdot n\otimes m_{\langle 1\rangle}\Big)\\
&\quad=\sum ((m_{(2)}\cdot n)_{\langle 1\rangle}\otimes m_{\langle 1\rangle\langle 1\rangle})\otimes (m_{(2)}\cdot n)_{(2)}m_{\langle 1\rangle(2)}\\
&\quad=\sum ((m_{(2)(2)}\cdot n)_{\langle 1\rangle}\otimes m_{\langle 1\rangle})\otimes (m_{(2)(2)}\cdot n)_{(2)}m_{(2)(1)}\\
&\quad=\sum (m_{(2)(1)}\cdot n_{\langle 1\rangle}\otimes m_{\langle 1\rangle})\otimes m_{(2)(2)}n_{(2)}\\
&\quad=\sum (m_{\langle 1\rangle(2)}\cdot n_{\langle 1\rangle}\otimes m_{\langle 1\rangle\langle 1\rangle})\otimes m_{(2)}n_{(2)}\\
&\quad=(\sigma_{M,N}\otimes I)(\rho(m\otimes n))\,,
\end{aligned}$$

it follows that $\sigma_{M,N}$ is a comodule map $\square$.

Let $R: M\otimes M\to M\otimes M$ be a solution to the quantum Yang–Baxter equation, where M is a finite-dimensional free k-module. Applying Theorem 11 to ${}_{A(R)}\mathcal{YD}^{A(R)}$ we see there are three braided monoidal categories naturally associated to R.

More generally, we make the definition of left invertible, right invertible, and invertible for any prebraided category $\mathscr{C}$ [18, Definition 1.12]. The subcategories $\mathscr{C}^{\triangleright}$, $\mathscr{C}^{\triangleleft}$ and $\mathscr{C}^{\bullet}$ are braided full subcategories.

When A has an antipode, the prebraiding structure on ${}_A\mathcal{YD}^A$ determined by the $\sigma_{M,N}$'s is a braiding structure by Lemma 1. When A^{cop} is a Hopf algebra, the prebraiding structure on ${}_A^A\mathcal{YD}$ determined by the $s_{M,N}$'s is a braiding structure by [18, Theorem 7.2]. For a particular bialgebra, whether or nor these prebraiding structures are both braiding structures or neither are is an interesting question. It can be the case that A is a Hopf algebra but A^{cop} is not [16].

The prebraiding morphisms $s_{M,N}$ defined in [18] for ${}_A^A\mathcal{YD}$ induce prebraiding structures on ${}_A\mathcal{YD}^A$, ${}^A\mathcal{YD}_A$ and $\mathcal{YD}_A^A$, respectively, under the identifications ${}_B^B\mathcal{YD}={}_A\mathcal{YD}^A$, ${}_C^C\mathcal{YD}={}^A\mathcal{YD}_A$ and ${}_D^D\mathcal{YD}=\mathcal{YD}_A^A$, where $B=A^{\mathrm{cop}}$, $C=A^{\mathrm{op}}$ and $D=A^{\mathrm{op\,cop}}$. The categorical isomorphisms of part (a) of Proposition 6 preserve the prebraiding structures, whereas the isomorphism of part (b) generally does not.

There is a second way of turning the tensor product of two objects M and N of ${}_A\mathcal{YD}^A$ into an object in ${}_A\mathcal{YD}^A$, which is derived from [7, Proposition 4.3.1]; namely by defining

$$a\bullet(m\otimes n)=\sum a_{(1)}\cdot m\otimes a_2\cdot n$$

and

$$\boldsymbol{\rho}(m\otimes n)=\sum (m_{\langle 1\rangle}\otimes n_{\langle 1\rangle})\otimes n_{(2)}m_{(2)}$$

for $a \in A$, $m \in M$ and $n \in N$. Note the 'twist' in the definition of the comodule structure on $M \otimes N$. With this notion of $\otimes$ we can put a second monoidal structure on ${}_A\mathcal{YD}^A$, using the $\mathbf{I}$, α, ρ and λ defined in Section 5. The k-linear maps $\boldsymbol{\sigma}_{M,N} : M \otimes N \to N \otimes M$ defined by

$$\boldsymbol{\sigma}_{M,N}(m \otimes n) = \sum n_{\langle 1 \rangle} \otimes n_{(2)} \cdot m$$

for $m \in M$ and $n \in N$ satisfy (B1) and (B2), and thus give a prebraiding structure to ${}_A\mathcal{YD}^A$, and hence a braiding structure to the analogs of $({}_A\mathcal{YD}^A)^{\triangleright}$, $({}_A\mathcal{YD}^A)^{\triangleleft}$ and $({}_A\mathcal{YD}^A)^{\bullet}$. Note that $\boldsymbol{\sigma}$ and σ are related by a 'double twist':

$$\boldsymbol{\sigma}_{M,N} = T_{M,N} \circ \sigma_{N,M} \circ T_{M,N} .$$

Let k be a field and M, N be finite-dimensional objects of ${}_A\mathcal{YD}^A$. Then both $\sigma_{M,N}$ and $\boldsymbol{\sigma}_{M,N}$ are manifestations of the same formula as we now show. Let A° be the dual bialgebra. Then M has a right A°-comodule structure (M, ρ°) such that $a \cdot m = (I \otimes a)(\rho^{\circ}(m))$ for all $a \in A$ and $m \in M$. The right A-comodule action of A on M induces a left A°-module structure on M defined by $a^{\circ} \bullet m = \sum m_{\langle 1 \rangle} a^{\circ}(m_{(2)})$ for $a^{\circ} \in A^{\circ}$ and $m \in M$. Let $\mathbf{M}$ denote the ambient vector space of M with these A°-structures.

Proposition 12. *Suppose that A is a bialgebra over a field k. Let M, N be objects of ${}_A\mathcal{YD}^A$ and $\mathbf{M}, \mathbf{N}$ be defined as above. Then*:

(a) *$\mathbf{M}, \mathbf{N}$ are objects of ${}_{A^{\circ}}\mathcal{YD}^{A^{\circ}}$*,

(b) $\boldsymbol{\sigma}_{M,N} = \sigma_{\mathbf{M},\mathbf{N}}$.

Proof. Part (a) is part (a) of [7, Theorem 7.3]. Part (b) follows by the calculations of part (b) of the same. □

Suppose that A is a finite-dimensional Hopf algebra over a field k, and let $(D(A), \mathcal{R})$ be the Drinfel'd double as defined in [2]. Write $\mathcal{R} = \sum \mathcal{R}^{(1)} \otimes \mathcal{R}^{(2)} \in A \otimes A$ and let M, N be left $D(A)$-modules. Then the map(s) $\mathcal{R}_{M,N} : M \otimes N \to N \otimes M$ defined by $\mathcal{R}_{M,N}(m \otimes n) = \sum \mathcal{R}^{(2)} \cdot n \otimes \mathcal{R}^{(1)} \cdot m$ for $m \in M$ and $n \in N$ give ${}_{D(A)}\mathcal{M}$ the structure of a braided monoidal category. By [13, Theorem 1] there is a categorical isomorphism ${}_{D(A)}\mathcal{M} \simeq {}_A\mathcal{YD}^A$ under which $\mathcal{R}_{M,N}$ and $\boldsymbol{\sigma}_{M,N}$ correspond. The reader should also see [12] at this point.

The $\sigma_{M,M}$'s and $\boldsymbol{\sigma}_{M,M}$'s are intertwined by braid relations. More generally, we have the following:

Proposition 13. *Suppose that $R : V \otimes V \to V \otimes V$ is a solution to* (4), *where V is a module over a commutative ring k. Set $L = T_V \circ R \circ T_V$. Then L is a solution to* (4) *and*

(a) $R_{23} \circ L_{13} \circ R_{23} = L_{13} \circ R_{23} \circ L_{13}$ *and* $L_{23} \circ R_{13} \circ L_{23} = R_{13} \circ L_{23} \circ R_{13}$,

(b) $R_{13} \circ L_{12} \circ R_{13} = L_{12} \circ R_{13} \circ L_{12}$ *and* $L_{13} \circ R_{12} \circ L_{13} = R_{12} \circ L_{13} \circ R_{12}$.

Proof. We have noted that L is a solution to (4). Let $u,v \in V$. We adapt the Heyneman–Sweedler notation for coalgebras and express the sum $R(u \otimes v)$ formally as $R(u \otimes v) = \sum u_{(1)} \otimes v_{(2)}$. Observe that $L(u \otimes v) = \sum u_{(2)} \otimes v_{(1)}$. In this notation (4) has the unambiguous formulation

$$\sum u_{(1)} \otimes v_{(1)(2)(1)} \otimes w_{(2)(2)} = \sum u_{(1)(1)} \otimes v_{(2)(1)(2)} \otimes w_{(2)} \tag{11}$$

for $u,v,w \in V$.

Since $R = T_V \circ L \circ T_V$ we need only establish the first two equations of parts (a) and (b). Now let $u,v,w \in V$. The equation $R_{23} \circ L_{13} \circ R_{23} = L_{13} \circ R_{23} \circ L_{13}$ follows from the calculations

$$\begin{aligned}
&R_{23} \circ L_{13} \circ R_{23}(u \otimes v \otimes w) \\
&\quad = R_{23} \circ L_{13}\left(\sum u \otimes v_{(1)} \otimes w_{(2)}\right) \\
&\quad = R_{23}\left(\sum u_{(2)} \otimes v_{(1)} \otimes w_{(2)(1)}\right) \\
&\quad = \sum u_{(2)} \otimes v_{(1)(1)} \otimes w_{(2)(1)(2)}
\end{aligned}$$

and

$$\begin{aligned}
&L_{13} \circ R_{23} \circ L_{13}(u \otimes v \otimes w) \\
&\quad = L_{13} \circ R_{23}\left(\sum u_{(2)} \otimes v \otimes w_{(1)}\right) \\
&\quad = L_{13}\left(\sum u_{(2)} \otimes v_{(1)} \otimes w_{(1)(2)}\right) \\
&\quad = \sum u_{(2)(2)} \otimes v_{(1)} \otimes w_{(1)(2)(1)} .
\end{aligned}$$

The equation $R_{13} \circ L_{12} \circ R_{13} = L_{12} \circ R_{13} \circ L_{12}$ follows in the same manner. $\square$

7. Invertibility of $\sigma_{M,N}$ when M and N are finite-dimensional

Let A be a bialgebra over a field k. In this section consider the question of when $\sigma_{M,N}$ is invertible for finite-dimensional objects M and N of ${}_A\mathcal{YD}^A$. In Theorem 16 we find sufficient conditions in terms of A. We find necessary and sufficient conditions terms of finite-dimensional simple objects of ${}_A\mathcal{YD}^A$ in Proposition 18. We also construct several examples, including one which shows that the categories $({}_A\mathcal{YD}^A)^{\triangleright}$, $({}_A\mathcal{YD}^A)^{\triangleleft}$ and $({}_A\mathcal{YD}^A)^{\bullet}$ are different in general.

Our first example, which is quite simple, will be used in the proof of Theorem 16.

Example 14. Let A be any cocommutative bialgebra over k and suppose that $g \in G(A)$ and is central. Then any left A-module $(M, \cdot)$ is the module structure of an object $(M, \cdot, \rho)$ of ${}_A\mathcal{YD}^A$, where ρ is defined by $\rho(m) = m \otimes g$ for all $m \in M$.

If A is a Hopf algebra, then $\sigma_{M,N}$ is an invertible linear map by Lemma 1. Our second example shows that A need not be a Hopf algebra in order that $\sigma_{M,N}$ be invertible for all finite-dimensional M, N.

Suppose that $g \in G(A)$ and set $M^g = \{m \in M \mid \rho(m) = m \otimes g\}$. Then M^g is a subcomodule of (M, ρ). Since distinct grouplike elements of a coalgebra over a field form a linearly independent set by part (b) of [15, Proposition 3.2.1], it follows that the sum $\sum_{g \in G(A)} M^g$ is direct. When A is spanned by grouplike elements it follows that $M = \bigoplus_{g \in G(A)} M^g$ by the same result.

Observe that

$$\sigma_{M,N}(m \otimes n) = \pi(g)(n) \otimes m \tag{12}$$

for $n \in N$ and $m \in M^g$, where $\pi : A \to \mathrm{End}_k(N)$ is the representation afforded by the left A-module structure on N.

Example 15. Let k be a field and $A = k\{x, y\}/(xy - 1)$ be the quotient of the free k-algebra on x and y modulo the ideal generated by $xy - 1$. Then A has a unique bialgebra structure such that x and y are grouplike elements. $\sigma_{M,N}$ is invertible for all finite-dimensional objects of ${}_A\mathcal{YD}^A$, but A is not a Hopf algebra.

To see this, let A be the bialgebra of Example 15, and suppose that M and N are finite-dimensional objects of ${}_A\mathcal{YD}^A$. Then $M = \bigoplus_{g \in G(A)} M^g$ since the coalgebra A is spanned by grouplike elements. Let $\pi : A \to \mathrm{End}_k(N)$ be the representation afforded by the module structure on N. Then $\pi(x) \circ \pi(y) = I$, which means that $\pi(x)$ and $\pi(y)$ are invertible. Thus $\pi(g)$ is invertible for $g \in G(A)$. Therefore, $\sigma_{M,N}$ is invertible by (12). But A is not a Hopf algebra since the grouplike element x is not invertible.

If, however, A is a finitely generated commutative cocommutative bialgebra over a field, then we have the following result:

Theorem 16. *Suppose that A is a commutative cocommutative bialgebra over a field k. Suppose further that A is finitely generated as an algebra. Then the following are equivalent:*

(a) *A is a Hopf algebra.*
(b) *$\sigma_{M,N}$ is invertible for all finite-dimensional objects M and N of ${}_A\mathcal{YD}^A$.*
(c) *$\sigma_{M,M}$ is invertible for all finite-dimensional objects M of ${}_A\mathcal{YD}^A$.*

Proof. In light of Lemma 1 we need only show that part (c) implies part (a). By [16, Corollary 69] we need only show that grouplike elements of A are invertible.

Assume the hypothesis of part (c). Let $g \in G(A)$ be fixed, and let $(M, \cdot)$ be a left A-module. We note that M has the structure of an object $(M, \cdot, \rho)$ of ${}_A\mathcal{YD}^A$ by Example 14, where $\rho(m) = m \otimes g$ for $m \in M$. Let $\pi : A \to \mathrm{End}_k(M)$ be the representation afforded by $(M, \cdot)$. Then by (12) we see that $\pi(g)$ is invertible when M is finite-dimensional.

Suppose that g is not invertible. Then g is contained in a maximal ideal I of A. Now $M = A/I$ is a field which is finitely generated as a k-algebra. Therefore, M is a finite-dimensional vector space over k by the Weak Nullstellensatz [5, 0.3]. This means that $I = A$, a contradiction. Hence g is invertible, and the theorem is proved. □

Since sub-bialgebras of commutative cocommutative Hopf algebras are not necessarily sub-Hopf algebras (for example the semigroup algebra $k[\mathbb{N}]$ of the group algebra $k[\mathbb{Z}]$), by Theorem 16 the property that $\sigma_{M,N}$ is invertible for all finite-dimensional objects M, N of ${}_A\mathcal{YD}^A$ is not hereditary.

Let M, N be finite-dimensional objects of ${}_A\mathcal{YD}^A$. It can be the case that $\sigma_{M,N}$ is invertible whenever M, N are cosemisimple (meaning they are the sum of their simple subcomodules), but generally $\sigma_{M,N}$ is not invertible.

Example 17. Let A be the algebra over a field k generated by symbols g and x subject to the relations

$$gx = -xg \quad \text{and} \quad x^2 = 0\,.$$

Then A has a unique bialgebra structure determined by

$$\Delta(g) = g \otimes g\,, \qquad \varepsilon(g) = 1\,,$$
$$\Delta(x) = x \otimes g + 1 \otimes x\,, \qquad \varepsilon(x) = 0\,.$$

Let M, N be finite-dimensional objects of ${}_A\mathcal{YD}^A$. Then $\sigma_{M,N}$ is invertible whenever M and N are cosemisimple, but $\sigma_{M,M}$ is not invertible for some 2-dimensional object M of ${}_A\mathcal{YD}^A$.

To verify the assertions of Example 17, we first establish that the simple subcoalgebras of A are kg^l for $l \geq 0$. To do this, we note that the simple subcoalgebras of $C = \mathrm{sp}(1, g, x)$ are $k1$ and kg. Let C_n be the nth term of the coradical filtration of C. Then $C_0 = \mathrm{sp}(1, g)$, and $C_{(n)} = \sum_{i_1 + \cdots + i_m \leq n} C_{i_1} \cdots C_{i_m}$ defines a filtration of A, since C generates A as an algebra. Therefore, $A_0 \subseteq C_{(0)}$ by [15, Proposition 11.1.1], and hence $A_0 = C_{(0)}$. It now follows that the simple subcoalgebras of A are kg^l for $l \geq 0$ by part (a) of [15, Proposition 8.0.3].

Let M and N be objects of ${}_A\mathcal{YD}^A$, and suppose that $\pi : A \to \mathrm{End}_k(N)$ is the representation afforded by the module structure on N. Note (6) can be expressed symbolically as

$$(\Delta)(a))(\rho(n)) = \sum \rho(a_{(2)} \cdot n)(1 \otimes a_{(1)}) \tag{13}$$

for $a \in A$ and $n \in N$. Now suppose that M and N are cosemisimple as comodules. Then $N = \bigoplus_{h \in G(A)} N^h$ since $\rho(N) \subseteq N \otimes A_0$. Now let $h \in G(A)$. Observe that $h = g^l$ for a unique $l \geq 0$, which we denote by $|h|$. From the relation $gx = -xg$ we derive $hx = (-1)^{|h|} xh$. Now let $n \in N^h$. Applying (13) to $g = a$ and n we deduce that $g \cdot n \otimes gh = \rho(g \cdot n)(1 \otimes g)$, or equivalently that $(g \cdot n \otimes h - \rho(g \cdot n))(1 \otimes g) = 0$. For $a \in A$ it is easy to see that $ag = 0$ implies that $a = 0$. Thus $\rho(g \cdot n) = g \cdot n \otimes h$. Now applying (13) to $x = a$ and n it follows that $\rho(x \cdot n) = x \cdot n \otimes gh - (g \cdot n - (-1)^{|h|} n) \otimes hx$. Therefore, $g \cdot n - (-1)^{|h|} n = 0$ since $\rho(N) \subseteq N \otimes A_0 = M \otimes \mathrm{sp}(1, g, g^2, \ldots)$. We have shown that $\pi(g)$ is invertible, and consequently $\pi(h)$ is invertible for all $h \in G(A)$. Since M is cosemisimple, it follows that $M = \bigoplus_{h \in G(A)} M^h$ as well; thus $\sigma_{M,N}$ is invertible by (12).

We now construct a 2-dimensional object of ${}_A\mathcal{YD}^A$ such that $\sigma_{M,M}$ is not invertible. Let M be a vector space over k with basis $\{m, n\}$. Then $\pi(g) = 0$, $\pi(x)(m) = n$ and $\pi(x)(n) = 0$ determines a representation $\pi : A \to \mathrm{End}_k(M)$. Note that $\rho(m) = m \otimes 1$ and $\rho(n) = n \otimes g + m \otimes x$ determines a right A-comodule structure on M. Since the set of all $a \in A$ such that (13) holds for all $m \in M$ is a subalgebra of A, it is easy to see that $(M, \cdot, \rho)$ is an object of ${}_A\mathcal{YD}^A$. But $\sigma_{M,M}$ is not invertible since $\sigma_{M,M}(n \otimes n) = 0$.

Notice that the 2-dimensional example M of the previous paragraph is a simple object of ${}_A\mathcal{YD}^A$, meaning that M has no proper subspaces where are at the same time submodules and subcomodules. The question of invertibility of all $\sigma_{M,N}$ in the finite-dimensional case comes down to invertibility of $\sigma_{M,N}$ when M and N are simple.

Proposition 18. *Suppose that A is a bialgebra over a field k. Then the following are equivalent*:

(a) $\sigma_{M,N}$ *is invertible for all finite-dimensional objects M, N of* ${}_A\mathcal{YD}^A$.

(b) $\sigma_{M,N}$ *is invertible for all finite-dimensional simple objects M, N of* ${}_A\mathcal{YD}^A$.

Proof. We need only show that part (b) implies part (a). Suppose that M and N are finite-dimensional objects of ${}_A\mathcal{YD}^A$. Let L be a subobject of M. Let $\pi : M \to M/L$ be the projection and $\iota : L \to M$ be the inclusion. Since L, M and N are comodules, and π and ι are comodule maps, we have the commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & L \otimes N \arrow[r, "\iota \otimes I"] \arrow[d] & M \otimes N \arrow[r, "\pi \otimes I"] \arrow[d] & M/L \otimes N \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & N \otimes L \arrow[r, "I \otimes \iota"] & N \otimes M \arrow[r, "I \otimes \pi"] & N \otimes M/L \arrow[r] & 0
\end{tikzcd}$$

where the rows are exact and the vertical arrows are the maps $\sigma_{L,N}$, $\sigma_{M,N}$ and $\sigma_{M/L,N}$ respectively. Therefore, if $\sigma_{L,N}$ and $\sigma_{M/L,N}$ are bijective then $\sigma_{M,N}$ is bijective.

Now suppose that L is a subobject of N. Then the projection $\pi : N \to N/L$ and the inclusion $\iota : L \to N$ are module maps. By a similar argument it follows that $\sigma_{M,N}$ is bijective if $\sigma_{M,L}$ and $\sigma_{M,N/L}$ are. Therefore, part (b) implies part (a) by induction on dim M + dim N, and the proof is complete. □

If $\sigma_{M,M}$ and $\sigma_{N,N}$ are invertible, then it may not be the case that $\sigma_{M\otimes N,M\otimes N}$ is invertible. We leave the details of the following example, which are easily verified, to the reader.

Example 19. Let A be a bialgebra over a field k and suppose $x, y \in G(A)$. Suppose that M, N are finite-dimensional non-zero objects of ${}_A\mathcal{YD}^A$ such that $M^x = x \cdot M = M$ and $N^y = y \cdot N = N$. If $y \cdot M = (0)$ or $x \cdot N = (0)$ then $\sigma_{M,M}$ and $\sigma_{N,N}$ are invertible but $\sigma_{M\otimes N,M\otimes N} = 0$.

The braided categories $({}_A\mathcal{YD}^A)^{\triangleright}$, $({}_A\mathcal{YD}^A)^{\triangleleft}$ and $({}_A\mathcal{YD}^A)^{\bullet}$ are not the same in general. Suppose that S is a commutative semigroup and let $A = k[S]$ be the semigroup algebra of S over a field k. Recall that any left A-module $(M, \cdot)$ can be turned into an object of ${}_A\mathcal{YD}^A$ by defining $\rho(M) = M \otimes h$ for any fixed $h \in G(A) = S$. Let M, N be objects of ${}_A\mathcal{YD}^A$, and write $M = \bigoplus_{g\in S} M^g$. Then $\sigma_{M,N}(M^g \otimes N) \subseteq g \cdot N \otimes M^g$. Therefore, M is left invertible if and only if left multiplication $g\cdot : N \to N$ is invertible for all $g \in S$ such that $M^g \neq (0)$. Likewise M right invertible if and only if $h\cdot : M \to M$ is invertible for all $h \in S$.

Example 20. Let $S = \{1, e\}$ be the semigroup with two elements, where 1 is the neutral element and e is an idempotent, and let $A = k[S]$ be the bialgebra described above. Now let M be a one-dimensional left A-module with basis m. Then $e \cdot m = 0$ and $\rho(m) = m \otimes 1$ give M the structure of a left invertible object of ${}_A\mathcal{YD}^A$ which is not right invertible. Likewise $e \cdot m = m$ and $\rho(m) = m \otimes e$ give M the structure of a right invertible object of ${}_A\mathcal{YD}^A$ which is not left invertible.

Our next example has an interesting asymmetry with respect to invertibility.

Example 21. Let A be the bialgebra of Example 17, and let $M = km$ be a one-dimensional vector space over the field k. Then $(M, \cdot, \rho)$ is an object of ${}_A\mathcal{YD}^A$, where

$$g \cdot m = m\,, \quad x \cdot m = 0 \quad \text{and} \quad \rho(m) = m \otimes g^2\,.$$

Observe that $a \cdot m = \varepsilon(a)$, for all $a \in A$. Therefore, $\sigma_{N,M} = T_{N,M}$ is invertible for any object N in ${}_A\mathcal{YD}^A$. Hence M is right invertible. There exists a 2-dimensional object N in ${}_A\mathcal{YD}^A$ such that $g \cdot N = (0)$ by Example 17. In this case $\sigma_{M,N} = 0$. Therefore, M is not left invertible. However all left invertible objects of ${}_A\mathcal{YD}^A$ are trivial, hence invertible.

We establish the assertions of the last sentence and leave the remaining details of the example to the reader. Suppose that M, N are objects of ${}_A\mathcal{YD}^A$, where A is any bialgebra over a field k. Suppose further that M is left invertible and that N is finite-dimensional. If L is a finite-dimensional subcomodule of M, it is easy to see that $\sigma_{L,N}$ is invertible, and hence $\sigma_{L/I,N}$ is invertible for any subcomodule I of L. Take I to be maximal and suppose that A is the bialgebra of Example 21. Then $\rho(L/I) = L/I \otimes h$ for some $h \in G(A)$ since L/I is simple. Taking N to be a two-dimensional object of ${}_A\mathcal{YD}^A$ such that $g \cdot N = (0)$, we deduce that $h = 1$. Since A has no non-zero primitive elements, we conclude by a simple induction that $\rho(L) = L \otimes 1$. Therefore, $\rho(M) = M \otimes 1$. By (13) we calculate $g \cdot m = m$ and $x \cdot m = 0$ for all $m \in M$. Therefore, $a \cdot m = \varepsilon(a)m$ and $\rho(m) = m \otimes 1$ for all $m \in M$. Therefore $(M, \cdot, \rho)$ is a trivial object, which means that it is invertible.

Finding necessary and sufficient conditions on a bialgebra A such that $\sigma_{M,N}$ is invertible for all finite-dimensional objects M, N of ${}_A\mathcal{YD}^A$ remains an open question. A question for further investigation is whether $\sigma_{M,N}$ invertible for all objects M, N of ${}_A\mathcal{YD}^A$ is enough to force A to be a Hopf algebra.

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